ACTUARIAL SOCIETY EXAMINATIONS IN 1905

QUESTIONS AND SOLUTIONS

REPRINTED FROM RECENT ISSUES OF

THE AMERICAN UNDERWRITER

AND

THE FUNDAMENTAL PRINCIPLES OF PROBABILITY

BY

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PREFACE

MR. ROBERT HENDERSON, the author of the solutions of Actuarial Society examination questions of 1905, presented in this book, and of the accompanying essay on Probability, was born May 24th, 1871, at Russell, near Ottawa, Canada. He was graduated from Toronto University, with the degree of B. A. in 1891, and was a University Fellow in mathematics for one year. In 1892, he went into the Dominion Insurance Department, Ottawa, Canada, under the Actuary of the Department, Mr. A. K. Blackadar, F. I. A., F. A. S. He remained until July, 1897, when he became associated with the Actuarial Department of the Equitable Life Assurance Society of the United States, of which Society he was appointed Assistant Actuary in September, 1903.

In 1896, Mr. Henderson took his final examination and became a Fellow of the Institute of Actuaries. He was enrolled as an Associate of the Actuarial Society in 1900, and was admitted on examination as a Fellow in 1902. By appointment of the President of the Actuarial Society, he was a member of the Committee on Examination for the years 1903, 1904 and 1905, the last year as Chairman. At the annual meeting of the Society, May 18, 1905, he was elected a member of the Council to serve for three years. To the *Transactions* of the Actuarial Society, his principal contributions have been, "Frequency Curves and Moments",

May 19, 1904, Vol. VIII, pp. 30-42; and "Note on Limit of Risk", May 18, 1905, Vol. IX, pp. 40-46.

Mr. Henderson has given the solutions of the questions in the examination of 1905 of the Actuarial Society in a clear, concise and logical manner. His essay on "The Fundamental Principles of Probability" is logical rather than mathematical in character and is written along the lines of the most recent treatises. In the belief that these solutions and this essay will be interesting and useful to many persons they are put in permanent form in this book. The utility of the book is enhanced by the fact that the left hand pages are blank for the convenience of those who desire to make additions, criticisms or annotations as to the particular subjects under consideration.

THE AMERICAN UNDERWRITER

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INTRODUCTION

THE following solutions of the questions proposed in the Actuarial Society's examination of 1905 were originally prepared for publication in THE AMERICAN UNDERWRITER and they are now brought together in more permanent form in the hope that they be of some assistance to students preparing for the examinations of the Society. In this connection the author considers it proper to point out, especially in view of his connection with the examination committee of that year, that the solutions given are intended only as specimens and not as the only appropriate answers to the questions.

Perhaps a few words as to the principles adopted in preparing these solutions may not be out of place. Where the questions referred to subjects taken up in text-books which were likely to be in the hands of students and others a mere reference was in most cases considered sufficient. With regard to questions in algebra—in particular—references are made to the text-book in which the author could most readily verify the references and to which he himself most frequently refers. The subjects of these questions will, however, be found taken up in any standard treatise on the subject and the student will refer to the particular treatise most convenient to himself.

In certain cases where an alternative treatment of a subject taken up in the text-books seemed worthy of notice it has been incorporated in the solution.

In the discussion of questions of a general nature the endeavor has been, so far as possible, to look at the questions from a fresh viewpoint. In all cases the answers have been made as concise as possible and such as might be written by a candidate working under a time limit.

A copy is included of the syllabus under which the examination was held. It is to be noted, however, that a change has been made in this syllabus which is intended to go into effect at the spring examinations in 1907. This change consists in adding to the final examination the subjects of Life Insurance Bookkeeping, Investments and Banking and Finance, and dividing the examination into two sections which may be taken in different years.

The opportunity is taken in collecting these solutions in book form to append a brief exposition of the fundamental principles of the theory of probability which was suggested by the two questions on that subject appearing in the examination papers. This essay is intended to show the connection of the mathematical theory of the subject with the logical theory as developed by the best modern writers on that subject.

ROBERT HENDERSON

ACTUARIAL SOCIETY
EXAMINATIONS IN 1905

SYLLABUS OF EXAMINATIONS

Associate

SECTION A

1. Arithmetic, elementary Algebra and the principles of double-entry Bookkeeping.
2. The following subjects in advanced Algebra:
   - *a.* Permutations and Combinations.
   - *b.* Binomial Theorem.
   - *c.* Series.
   - *d.* Theory and use of Logarithms.
   - *e.* The elements of Finite Differences, including interpolation and Summation.
3. The elements of the Theory of Probabilities.
4. Compound Interest and Annuities Certain.
5. Elementary plane Geometry.*
6. Practical examples in the foregoing subjects.

SECTION B

1. The application of the Theory of Probabilities to Life Contingencies.
2. Theory of Annuities and Assurances, including the theory and use of Commutation Tables and the computation of premiums for usual contract.
3. Valuation of ordinary forms of policies.
4. Practical examples on all the above, involving Joint as well as Single lives.

5. General nature of Insurance contracts.

6. The outines of the history of Life Insurance.

7. The source and characteristics of the principal Mortality Tables.

Fellow

1. Methods of constructing and graduating Mortality Tables and the use of the formulas of Gompertz and Makeham.

2. Methods of loading premiums to provide for expenses and contingencies.

3. Valuation of the liabilities and assets of Life Insurance companies.

4. The assesment of expenses and the distribution of surplus.

5. Practical treatment of cases of alteration or surrender of Life Insurance contracts.

6. Application of the Calculus of Finite Differences and of the Differential and Integral Calculus to Life Contingencies.

7. Laws of the United States and Canada relating to Life Insurance.

8. Insurance of Under-Average Lives and extra premiums for Special Hazards.

*Note for Canadian candidates. Euclid Books I, II and III will be considered sufficient for this subject.

ERRATA.

Page 23, Question 6, line 4 — Insert δ after word "computing."

Page 25, Question 8, line 2 — Read "annual" in place of "anual."

Question 10, line 4 — read numerator "2401."

Page 29, Solution 13 (a), line 4 — Make "$\frac{m}{p}$" read as index.

Solution 13 (a), line 5 — Make "n" curtate.

Page 31, Solution 14, next line to bottom — Insert "angle" before "P A B′."

Page 33, Solution 1, line 7 — Read "l_{x+1}" in place of "l 1."

Page 41, Solution 10, line 2, — Read ".02186 $\times$" before the fraction in place of ".02186 +."

Question 11, line 1 — Read A^1_{xyz}.

Solution 12, line 4 — Read "M_x" in place of "$M x$."

Page 43, Solution 14 (a), line 3 — Read "sixty" in place of "twenty-three."

Page 45, Solution 1, line 11 — Insert "by" after "existing."

Page 53, Solution 11, line 7 — Read "$(1+j)$" in place of "$(i+j)$."

Page 57, Solution 15, line 7 — Read "u^1_x" in place of "u^1."

Page 61, Question 22 (b), line 2 — Read "l_x" in place of "$l x$."

Page 91, Second paragraph, line 6 — Read "$n+1$" in place of "$n-1$."

ACTUARIAL SOCIETY EXAMINATIONS

SOLUTIONS OF QUESTIONS IN EXAMINATION FOR ADMISSION AS ASSOCIATE OR FELLOW, HELD APRIL 13 AND 14, 1905

N. B.—Actuarial Society or American notation

Associate—Section A

1. (*a*) *Simplify* $\dfrac{\frac{1}{3} - \frac{1}{5} + \frac{3}{16}}{\frac{1}{3} - \frac{1}{4} + \frac{2}{25}}$.

   (*b*) *Extract the square root of* 1,012,766,976.

   (*a*) $1.\frac{27}{28}$.

   (*b*) 31,824.

2. (*a*) *Prove that a quadratic equation can have only two roots.*

   (*b*) *Solve the equation.*

$$a^2 \frac{(x - b)\ (x - c)}{(a - b)\ (a - c)} + b^2 \frac{(x - c)\ (x - a)}{(b - c)\ (b - a)} = x^2$$

(*a*) Let α and β be two roots of the equation, then since $x - \alpha$ and $x - \beta$ are factors the equation must take the form $a\ (x - \alpha)\ (x - \beta) = o$. If, then, any other value y of x satisfies this equation we have $a\ (y - \alpha)\ (y - \beta) = o$ which is impossible unless $a = o$, in which case the equation would vanish identically. Therefore, the equation cannot have more than two roots.

(*b*) Substituting a for x in the equation it becomes

$$a^2 \frac{(a - b)\ (a - c)}{(a - b)\ (a - c)} + o = a^2$$

Since this is an identity a satisfies the equation, similarly b satisfies it and the equation does not vanish identically since $x = c$ does not satisfy it. As the equation is of the second degree and does not vanish identically it has only two roots and these we have seen to be a and b.

✓ 3. (*a*) *Given that the Binomial Theorem is true when the index is a positive integer, prove that it is also true for negative or fractional indices.*

(*b*) *Expand* $(1 - x)^{-2}$.

(*a*) For proof of first part of question refer to C. Smith's Algebra, 2d Edition, Article 283.

(*b*) By the Binomial Theorem.

$$(1 - x)^{-2} = 1 - \frac{(-2)}{1} x + \frac{(-2)\ (-3)}{1 \cdot 2} x^2 - \frac{(-2)(-3)(-4)}{1 \cdot 2 \cdot 3} x^3$$
$$+ \text{ etc.}$$
$$= 1 + 2x + 3x^2 + 4x^3 + \text{etc.}$$

✓ 4. *Expand* e^x *in ascending powers of* x.

Hence, where c_0, c_1, *etc., are the coefficients of the various powers of* x *in* $(1 + x)^n$ *prove that* $c_0\ (a + n)^m - c_1\ (a + n - 1)^m + c_2\ (a + n - 2)^m$ *—etc., vanishes if* m *is less than* n *and is equal to* $\underline{|n}$ *if* m *is equal to* n (*m and n being both integral*).

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \text{etc.}$$

For proof refer to C. Smith's Algebra, 2d Edition, Article 302.

$c_0\ (a + n)^m - c_1\ (a + n - 1)^m + c_2\ (a + n - 2)^m$ — etc., is $\underline{|m}$ times the coefficient of x^m in the expansion of $c_0\ e^{(a+n)\,x} - c_1\ e^{(a+n-1)\,x} + c_2\ e^{(a+n-2)\,x}$ — etc.

or, of $e^{ax}\ (e^x - 1)^n$.

But $(e^x - 1) = x + \frac{x^2}{\underline{|2}} + \text{etc.}$

$\therefore (e^x - 1)^n = x^n + \frac{n}{2} x^{n+1} + \text{etc.}$

and $e^{ax} = 1 + ax + \text{etc.}$

$\therefore e^{ax}\ (e^x - 1)^n = x^n + (a + \frac{n}{2})\ x^{n+1} + \text{etc.}$

So that the coefficient of all powers of x less than n is zero, and that of x^n is unity. Hence the proposition follows.

✓ 5. *How are probabilities measured? Give instances where addition, subtraction, multiplication and division of probabilities, respectively, give interpretable results, stating in each case the relation of the resulting probability to those operated upon.*

The measure of the probability of an event happening under given circumstances is the limit of the ratio of the number of times the event happens under the given circumstances, to the number of times the given circumstances occur, when the number of cases is indefinitely increased. The probabilities of two mutually exclusive events may be added together, the result giving the probability that either one or the other will happen. If two events be so related that the first cannot happen unless the second does, the probability of the first may be subtracted from that of the second, the result giving the probability that the second will happen unaccompanied by the first. The probability of an event may be multiplied by the chance that if it happens a second event will happen also, the result being the probability that both will happen. Conversely the probability of a compound event may be divided by the probability of one of its components, the result being the probability that if that event happens the other will happen also.

6. *If i is the effective annual rate of interest, j the nominal rate convertible m times a year, d the rate of discount and δ the force of interest; express each in a series of ascending powers of each of the others. Give an approximate method of computing δ for a given rate of interest.*

6. If i is the effective annual rate of interest, j the nominal rate convertible m times a year, d the rate of discount and δ the force of interest; express each in a series of ascending powers of each of the others. Give an approximate method of computing δ for a given rate of interest.

See Institute of Actuaries' Text Book, Part I, new edition, Chapter I, Article 30.

From the expressions for i and d in terms of δ we have

$$i + d = 2\delta + \frac{\delta^3}{3} + \text{etc.}$$

or approximately $\delta = \dfrac{i + d}{2}$

also $i\,d = \delta^2 + \dfrac{1}{12}\delta^4 + \text{etc.}$

so that approximately $\delta = \sqrt{i\,d}$.

7. *Having given the value of an annuity certain for a term of years, deduce a formula for finding the approximate rate of interest.*

Suppose a is the value of the annuity and n the period, and let i be the rate of interest. Let also a_1, near to a, be the value at some known rate i_1, then we have approximately $\dfrac{a - a_1}{i - i_1} = \dfrac{d\,a_1}{d\,i_1}$

but $i_1\,a_1 = 1 - (1 + i_1)^{-n}$

$$\therefore a_1 + i_1 \frac{d\,a_1}{d\,i_1} = n\,(1 + i_1)^{-n+1} = n\,v_1^{\,n+1}$$

$$\text{or } \frac{d\,a_1}{d\,i_1} = -\,\frac{a_1 - n\,v_1^{\,n+1}}{i_1}$$

$$\text{whence } i - i_1 = i_1\,\frac{a_1 - a}{a_1 - n\,v_1^{\,n+1}} \text{ or } i = i_1 + i_1\,\frac{a_1 - a}{a_1 - n\,v_1^{\,n+1}}$$

If desired this rate may be used as a basis for a further approximation.

8. *Define Revenue Account, Balance Sheet and Inventory.*

What items in the anual statement of a life insurance company are ordinarily the result of an inventory?

The Revenue Account shows on one side the amount of the policy reserve and the surplus at the beginning of the year and the income earned during the year, and on the other side the expenditure incurred during the year, the balance being the amount of the policy reserve and surplus at the end of the year.

The Balance Sheet is made up of the balance of the other accounts showing, on one side, the capital stock, if any, the policy reserve, the liabilities outstanding on the various accounts and the surplus, and on the other side the assets on hand

An Inventory is a computation of the total amount of a given kind of assets or liabilities made by valuing the individual items. Interest and rents due and accrued and market value of bonds and stocks over book value are ordinarily determined by inventory. The liability for reserve may also be considered as so determined.

✓ 9. *Find the value of ${}_nC_r$, the number of combinations of n different things taken r at a time.*

Prove that, if x and y be any two positive integers, then will

$${}_{x+y}C_n = {}_xC_n + {}_xC_{n-1}\,{}_yC_1 + {}_xC_{n-2}\,{}_yC_2 + \ldots\ldots + {}_yC_n.$$

See C. Smith's Algebra, 2d Edition, Articles 244 and 248.

✓ 10. *Given $\log_{10}2 = .30103$ and $\log_{10}3 = .47712$, calculate the logarithms of all numbers from 4 to 10 inclusive, using, where necessary, a finite difference formula.*

From these values calculate $\log_{10}\frac{2401}{2400}$.

We have

$$\begin{aligned}
\log 4 &= 2\log 2 &&= .60206 \\
\log 5 &= \log 10 - \log 2 &&= 1 - .30103 = .69897 \\
\log 6 &= \log 2 + \log 3 &&= .77815 \\
\log 8 &= 3\log 2 &&= .90309 \\
\log 9 &= 2\log 3 &&= .95424 \\
\log 10 &= &&1.00000
\end{aligned}$$

To determine log 7 we have

$$\begin{aligned}
\log 45 &= \log 5 + \log 9 = 1.65321 \\
\log 48 &= \log 6 + \log 8 = 1.68124 \\
\log 50 &= \log 5 + \log 10 = 1.69897 \\
\log 54 &= \log 6 + \log 9 = 1.73239
\end{aligned}$$

Whence interpolating by Lagrange's formula

$$\log 49 = -\tfrac{1}{27}\log 45 + \tfrac{5}{9}\log 48 + \tfrac{1}{2}\log 50 - \tfrac{1}{54}\log 54 = 1.69020$$

$$\log 7 = \tfrac{1}{2}\log 49 = .84510$$

Also we have $\log\frac{2401}{2400} = 4\log 7 - \log 3 - 3\log 2 - 2 = .00019$

11. (*a*) *Define central differences.*

(*b*) *Expand* $\dfrac{u_{x+n} + u_{x-n}}{2}$ *and* $\dfrac{u_{x+n} - u_{x-n}}{2}$ *in terms of* u_x *and its central differences.*

(*a*) Central differences are the differences of a function referred to the central value of the variable. As there is no value of the odd orders of differences corresponding to the values of the original function the mean of the two adjacent values is used.

(*b*) We have,

$$u_{x+n} = u_x + n\,a_x + \frac{n^2}{\underline{|2}}\,b_x + \frac{n\,(n^2-1)}{\underline{|3}}\,c_x + \frac{n^2\,(n^2-1)}{\underline{|4}}$$

$$d_x + \text{etc.}$$

$$u_{x-n} = u_x - n\,a_x + \frac{n^2}{\underline{|2}}\,b_x - \frac{n\,(n^2-1)}{\underline{|3}}\,c_x$$

$$+ \frac{n^2(n^2-1)}{\underline{|4}}\,d_x + \text{etc.}$$

where a_x, b_x, c_x, d_x, etc. are the successive central differences corresponding to u_x

$$\text{Whence } \frac{u_{x+n} + u_{x-n}}{2} = u_x + \frac{n^2}{\underline{|2}}\,b_x + \frac{n^2\,(n^2-1)}{\underline{|4}}\,d_x + \text{etc.}$$

$$\frac{u_{x+n} - u_{x-n}}{2} = n\,a_x + \frac{n\,(n^2-1)}{\underline{|3}}\,c_x + \text{etc.}$$

12. (*a*) *The probability of the happening of a certain event is p, and the probability of its failure is q; what is the probability of its happening exactly r times in n trials?*

(*b*) *For what value of r is this probability the greatest?*

(*c*) *Does this maximum probability increase or decrease as n increases?*

(*a*) The chance of the event happening on any particular r trials and failing on the remainder $n-r$ is $p^r q^{n-r}$. But the number of different combinations of r trials is $\dfrac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}$ so that the total probability is $\dfrac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}\,p^r\,q^{n-r}$

(*b*) This value is the greatest for the largest value of r for which

$$\frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}\,p^r\,q^{n-r} > \frac{\underline{|n}}{\underline{|r-1}\;\underline{|n-r+1}}\,p^{r-1}\,q^{n-r+1}$$

$(n-r+1)\,p > r\,q$, or $r\,(p+q) < (n+1)\,p$

or, since $p+q=1$, when r is the integral part of $(n+1)\,p$.

(c) Putting now $(n + 1)$ for n the maximum probability is either $\frac{|\underline{n+1}}{|\underline{r}\;|\underline{n-r+1}}\, p^r\, q^{n-r+1}$ or $\frac{|\underline{n+1}}{|\underline{r+1}\;|\underline{n-r}}\, p^{r+1}\, q^{n-r}$ the ratio of which to the preceding is either $\frac{(n+1)\,q}{n-r+1}$ or $\frac{(n+1)\,p}{r+1}$, both of which are less than unity since $(n + 1)\, p$ lies between r and $r + 1$ and consequently $(n + 1)\, q$ between $n - r$ and $n - r + 1$.

13. (a) *What is the value of an annuity certain for n years, payable p times per annum at the nominal rate j convertible m times a year.*

(b) *An annual annuity running n years is worth twenty-five year's purchase; one running 2n years is worth thirty years' purchase. What is the rate of interest?*

(a) We have where i is the effective rate of interest per annum:

$$a^{(p)}_{\overline{n|}} = \frac{1}{p}\left(v^{\frac{1}{p}} + v^{\frac{2}{p}} + \ldots\ldots\ldots\ldots v^n\right) = \frac{1 - v^n}{p\left[(1+i)^{\frac{1}{p}} - 1\right]}$$

but $1 + i = \left(1 + \frac{j}{m}\right)^m$, so that

$$a^{(p)}_{\overline{n|}} = \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{p\left[\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1\right]}$$

Where p is equal to m, this becomes $a^{(m)}_{\overline{n|}} = \frac{1 - v^n}{j}$

(b) If $a^{(m)}_{\overline{n|}} = \frac{1 - v^n}{j} = 25$

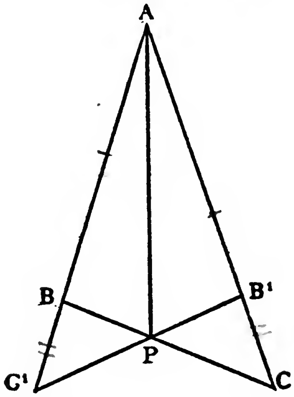
and $a^{(m)}_{\overline{2n|}} = \frac{1 - v^{2n}}{j} = 30$,

we have $\frac{a^{(m)}_{\overline{2n|}}}{a^{(m)}_{\overline{n|}}} = \frac{1 - v^{2n}}{1 - v^n} = 1 + v^n = \frac{6}{5}$

therefore $1 - v^n = \frac{4}{5} = 25j$, since $a^{(m)}_{\overline{n|}} = 25$

therefore $j = \frac{4}{125}$, or the nominal rate of interest, convertible with the same frequency as the annuity is payable, is $3\frac{1}{5}$ per cent.

✓ 14. *On the side* A B, *produced if necessary, of a triangle* A B C, A C′ *is taken equal to* A C; *similarly on* A C, A B′ *is taken equal to* A B, *and the line* B′ C′ *is drawn to cut* B C *in* P. *Prove that the line* A P *bisects the angle* B A C.



Let the construction be made as described and suppose C′ is in A B produced and consequently B′ in A C.

Then in the triangles A B C and A B′C′ since A B=A B′ A C= A C′ and angle B A C=angle B′ A C′ the triangles are equal in every respect, or angle A B C=angle A B′ C′, and angle A C B=angle A C′ B′

Also since angles C′ B P and C B′ P are supplementary to the angles A B C and A B′ C′ they are equal.

Also since A B = A B′ and A C′ = A C therefore B C′ = B′ C.

Then in the triangles P B C′ and P B′ C, B C′ = B′ C angle P B C′= angle P B′ C and angle P C′ B=angle P C B′ therefore P B = P B′.

Then in the triangles P A B and P A B′, P B = P B′ A B = A B′ and P A is common therefore angle P A B = P A B′ or P A bisects the angle B A C.

Associate—Section B

✓ 1. *What does the mortality table represent, and of what columns does it usually consist?*

Express in terms of the mortality table the probabilities:

(a) That one of the lives (x) and (y), will survive n years and the other will fail within n years.

(b) That at least one of the lives will fail in n years.

(c) That both will survive n years.

(d) That both will die within n years.

(e) That the first death will happen in the n^{th} year from the present time.

It represents the history as regards mortality of a representative group of people, and indicates the proportion which dies in each year of age and the proportion which survives to the end of each year. The mortality table proper consists of the column of l_x showing the proportion of those attaining the initial age which survives to each subsequent age, and ths column of d_x showing the proportion of the same group which dies in each year, so that necessarily $d_x = l_x - l_{x+1}$

(a) $${}_np_x(1 - {}_1p_y) + {}_np_y(1 - {}_np_x)$$
$$= {}_np_x + {}_np_y - 2\,{}_np_x \cdot {}_np_y$$
$$= \frac{l_{x+n}}{l_x} + \frac{l_{y+n}}{l_y} - 2\frac{l_{x+n} : l_{y+n}}{l_x \cdot l_y}$$

(b) $$1 - {}_np_{x} : {}_np_y = 1 - \frac{l_{x+n} : l_{y+n}}{l_x \cdot l_y}$$

(c) $${}_np_x : {}_np_y = \frac{l_{x+n} : l_{y+n}}{l_x : l_y}$$

(d) $$(1 - {}_np_x)(1 - {}_np_y) = \frac{(l_x - l_{x+n})(l_y - l_{y+n})}{l_x : l_y}$$
$$= 1 - \frac{l_{x+n}}{l_x} - \frac{l_{y+n}}{l_y} + \frac{l_{x+n} : l_{y+n}}{l_x : l_y}$$

(e) $${}_{n-1}p_x : {}_{n-1}p_y - {}_np_x \cdot {}_np_y$$
$$= \frac{l_{x+n-1} : l_{y+n-1} - l_{x+n} : l_{y+n}}{l_x \cdot l_y}$$

2. *Express the value of a_x in terms of the mortality table.*
Prove that $a_x < a_{\overline{e_x}|}$.

If annuities of a unit be issued on each of l_x lives aged x the total amount to be paid at the end of the first year is l_{x+1}, at the end the second year l_{x+2}, and so on. So that equating present values we have

$$l_x a_x = v\, l_{x+1} + v^2 l_{x+2} + \text{etc.}$$

$$\text{or } a_x = \frac{v\, l_{x+1} + v^2 l_{x+2} + \text{etc.}}{l_x}$$

We know that $A_x > v^{e_x + 1}$. For the former is the arithmetic and the latter is the geometric mean of the present values of the death claims. Therefore

$$a_x = \frac{1 - (1 + i)\, A_x}{i} < \frac{1 - v^{e_x}}{i}\text{; that is } < a_{\overline{e_x}|}.$$

3. *On De Moivre's hypothesis find the value of a life annuity in terms of the expectation of life.*
Does this formula give results sufficiently close for practical use?

On De Moivre's hypothesis, if n is the complement of life, we have, since the decrements are supposed uniform, $\mathring{e}_x = \frac{n}{2}$

Also, $A_x = \frac{a_{\overline{n}|}}{n}$ so that

$$1 + a_x = \frac{1 - A_x}{d} = \frac{n - a_{\overline{n}|}}{n\, d} = \frac{2\mathring{e}_x - a_{\overline{2\mathring{e}_x}|}}{2\mathring{e}_x \,.\, d}.$$

Owing to the lack of conformity of mortality tables with De Moivre's hypothesis this formula does not give results sufficiently close for practical purposes.

4. *Deduce formulas for the net single and annual premiums for a survivorship annuity payable to (x) after the death of (y).*

The payment under the survivorship annuity will be due at the end of any year if (x) be alive and (y) be dead. The net single premium for the benefit is therefore,

$$\Sigma\, v^n\, {}_np_x (1 - {}_np_y)$$

$$= \Sigma\, v^n\, {}_np_x - \Sigma\, v^n\, {}_np_x \cdot {}_np_y$$

$$= a_x - a_{xy}.$$

The annual premium is ordinarily payable during the joint lives so that denoting its value by π we have

$$\pi (1 + a_{xy}) = a_x - a_{xy}$$

$$\text{or } \pi = \frac{a_x - a_{xy}}{1 + a_{xy}} = \frac{1 + a_x}{1 + a_{xy}} - 1.$$

5. *Having given the values of* D_x, N_x *and* S_x, *show how the values of* C_x, M_x *and* R_x *may be ascertained.*

Find the annual premium for whole life insurance, all premiums paid to be returned at death, assuming that the premium is loaded by a percentage and a constant.

$$C_x = v^{x+1} d_x = v^{x+1}(l_x - l_{x+1}) = v\, D_x - D_{x+1}$$

$$M_x = \Sigma\, C_x = v\, \Sigma\, D_x - \Sigma\, D_{x+1} = v\, N_x - N_{x+1}$$

$$= D_x - d\, N_x$$

$$R_x = \Sigma\, M_x = \Sigma\, D_x - d\, \Sigma\, N_x = N_x - d\, S_x$$

Let π_1 = office premium required and π = net premium where

$$\pi_1 = (1 + k)\, \pi + c$$

Then $\pi\, N_x = M_x + R_x \left\{ (1 + k)\, \pi + c \right\}$

or $\pi \left\{ N_x - (1 + k)\, R_x \right\} = M_x + c\, R_x$

$$\pi = \frac{M_x + c\, R_x}{N_x - (1 + k)\, R_x}$$

$$\pi_1 = (1 + k)\, \pi + c = \frac{(1 + k)\, M_x + c\, N_x}{N_x - (1 + k)\, R_x} = \frac{(1 + k)\dfrac{M_x}{N_x} + c}{1 - (1 + k)\dfrac{R_x}{N_x}}$$

$$= \frac{(1 + k)\, P_x + c}{1 - (1 + k)\dfrac{R_x}{N_x}}$$

6. *Investigate the relation between the rates of mortality in two tables giving the same net premium reserves on ordinary life policies. What conclusion would you draw regarding the effect on such reserves of an increase in the rate of interest?*

We have for all values of n

$$\left\{ {}_{n-1}V_x + P_x \right\} (1 + i) = p_{x+n-1}\, {}_nV_x + q_{x+n-1} =$$

$${}_nV_x + q_{x+n-1}(1 - {}_nV_x).$$

Suppose any other table of mortality gives the same terminal reserve values and denote net premiums and rates of mortality according to this second table by accented letters then we have

$$\left\{ {}_{n-1}V_x + P^1_x \right\} (1 + i) = {}_nV_x + q^1_{x+n-1}(1 - {}_nV_x).$$

Subtracting we get $(P^1_x - P_x)(1 + i) = (q^1_{x+n-1} - q^1_{x+n-1})$

$(1 - {}_nV_x)$.

or $(P^1_x - P_x)(1 + i)(1 + a_x) = (q^1_{x+n-1} - q_{x+n-1})$

$(1 + a_{x+n})$.

for all values of n so that

$(q^1_x - q_x)(1 + a_{x+1})$ must be a constant

or q^1_x can be expressed in the form $q_x + \dfrac{k}{1 + a_{x+1}}$.

An increase in the rate of interest is equivalent so far as annuity values are concerned to a proportionate reduction of the values of p_x or, for adult lives, a decreasing addition to the value of q_x whereas in order to give the same reserves an increasing addition to q_x is required, consequently we should expect that an increase in the rate of interest would in general lower the reserves.

7. *State the leading points of difference between policies now issued in America and those issued fifty years ago.*

Fifty years ago policies were issued on comparatively few forms, principally ordinary life, the many special forms which are now such a feature of the business being then unknown. The many privileges and options as to surrender values, dividends, etc., now granted, were then absent from the contract which on the other hand contained many restrictions and prohibitions relating to travel, occupation, and residence. Deferred dividend policies were first issued in 1868.

8. *Find the values of* Q^1_{xy}; $Q^1_{x:\overline{y(\overline{t|})}}$ *and* $e_{x:\overline{y(\overline{t|})}}$.

Among those cases where (x) actually survives (y), what is the *average duration of the period of survival?*

See Institute of Actuaries' Text Book, Part 2, Chapter 4, Articles 3, 13 and 16.

In a large number N of cases the number where (x) survives (y) is $N\,Q^1_{xy}$ and the total number of years included in all the periods of such survival is $N\,\mathring{e}_{y|x} = N\,(e_x - e_{xy})$. Therefore, the average period is $\dfrac{e_x - e_{xy}}{Q^1_{xy}}$.

9. *From general reasoning find the value of* A_x *in terms of* a_x. *Express also its value in terms of the mortality table and in commutation symbols.*

See Institute of Actuaries' Text Book, Part II, Chapter VII, Articles 32, 33, 41, 42, 43 and 44.

10. *Give formulas for the ordinary life, limited payment life and endowment assurance annual premium rates in terms of annuity values and also in terms of commutation columns.*

Given $P_x = .02186$, $P_{x\overline{n|}} = .04220$ *and* $i = .03$; *find* ${}_nP_x$.

See Institute of Actuaries' Text Book, Part II, Chapter VII, Articles 61, 69, 73, 79 and 80.

We have $_nP_x = \dfrac{A_x}{1 + a_{x:\overline{n-1}|}} = A_x (P_{\overline{xn}|} + d) = \dfrac{P_x}{P_x + d}(P_{\overline{xn}|} + d)$

$$= P_x \frac{(1 + i)\, P_{\overline{xn}|} + i}{(1 + i)\, P_x + i} = .02186 \times \frac{1.03 \times .04220 + .03}{1.03 \times .02186 + .03}$$

$$= .02186 \times \frac{.0734660}{.0525158} = .03058$$

11. *Find an expression for* A^1_{xyz} *and show how to express in terms of simpler functions the values of* A^2_{xyz}, A^3_{xyz} *and* $A^1_{x:\overline{yz}}$.

See Institute of Actuaries Text Book, Part II, Chapter XIII, Articles 24, 26 and 27.

12. *Write down formulae for the value of a limited payment life policy by the prospective method and by the retrospective method, explaining each. State under what conditions they give identical results.*

Prospectively $_nV = \dfrac{M_{x+n} - \pi_x (N_{x+n} - N_{x+m})}{D_{x+n}}$ where m is the premium period.

Retrospectively $_nV = \dfrac{\pi_x (N_x - N_{x+n}) - (M_x - M_{x+n})}{D_{x+n}}$

The difference between these two values is $\dfrac{\pi_x (N_x - N_{x+m}) - M_x}{D_{x+n}}$

which vanishes if $\pi_x = \dfrac{M_x}{N_x - N_{x+m}}$, that is, if the premium valued is the net premium for the policy on the basis of valuation. Otherwise, the two will give different results, unless the retrospective formula be properly modified.

13. *State as briefly as possible the main points to be covered in drawing up a policy of insurance on the limitedpayment life plan.*

The special points to be covered in drafting a limited-payment life policy, as distinguished from other forms, are that the face of the policy is payable on receipt of satisfactory proofs of death of the assured and that the assurance is conditional on the payment of the specified premium annually in advance during a specified period or until the prior death of the assured. These are the distinguishing features of a limited payment life policy, the other privileges and conditions being in general common to this and other forms. The conditions embody such restrictions and limitations regarding oc-

cupation, travel, residence, suicide, and proof of age, as may be considered advisable, and usually incorporate by reference the application as part of the contract. The privileges refer to such features as incontestability, grace in payment of premiums, reinstatement, change of beneficiary, loans and surrender values.

14. (*a*) *What are the leading features which distinguish the* O^M *table from the Actuaries' table?*

(*b*) *What was found to be the experience of policies with and without profits respectively and of endowments as compared with whole-life policies?*

(*a*) As regards source the O^M table is distinguished by being the result of the recent experience of a large and homogeneous class of risks, being the experience of ~~twenty-three~~ sixty British offices during the years 1863 to 1893 on male lives assured under ordinary life policies, while the Actuaries' table is based on the whole experience of seventeen offices up to 1839. As regards methods, the O^M table was made up on lives, while the Actuaries' table was made up on policies, except that in some companies duplicates in the same company were eliminated. As regards results, the O^M table shows a relatively very low rate of mortality at the younger ages as compared with the Actuaries' table. The difference in rates of mortality becomes relatively very small as the age advances. The expectation by the O^M table is about five per cent. higher at age 20 than by the Actuaries' table. At age 35 this difference is reduced to about two and one half per cent. and remains at that figure beyond age 70.

(*b*) In the British offices experience, the mortality on lives assured with profits was much lighter than on those assured without profits, and the experience on endowments was much more favorable than on whole life policies.

Fellow—

1. Draft a memorandum of instructions to be followed by clerks in taking out the mortality experience of a company, the result to be exhibited in the form of select tables.

From proper record enter on cards the age at entry, date of issue and date and mode of termination of risk (if terminated). Include only terminations occurring before the anniversary in the year 1904. In the case of terminations otherwise than by death, record, as the duration, the difference between the calendar years of entry and exit. In the case of deaths add one year to the number of complete policy years which have elapsed between date of issue and date of death. Take the age at entry at nearest birthday. Sort cards into deaths, other terminations (withdrawals) and existing, then sort each lot by age at entry. Sort the deaths and withdrawals at each age according to duration, and the existing by calendar year of issue (the duration being the difference between that year and the year of termination of observations). Record on a schedule for each age at entry the total number for each duration, separately for the deaths, withdrawals and existing. (The cards should contain further information such as kind of policy, residence, etc., if required for the purposes of subdivision).

2. On the assumption of Makeham's law, find the value of $\bar{A}^{1}_{xy}$ in terms of the constants and $\bar{A}_{xy}$. Hence give approximately the value of A^{1}_{xy} in terms of A_{xy}.

We have by Makeham's law,

$$\mu_{x+t} = A + Bc^{x+t}$$

and $\mu_{y+t} = A + Bc^{y+t}$

where $A = -\log_e s$.

$$\therefore c^{y}(\mu_{x+t} + \log_e s) = c^{x}(\mu_{y+t} + \log_e s)$$

multiplying then through by

$v^{t}\,{}_{t}p_{xy}$ and integrating with respect to t, we get

$$c^{y}(\bar{A}^{1}_{xy} + \log_e s\,\bar{a}_{xy}) = c^{x}(\bar{A}_{x\overset{1}{y}} + \log_e s\,\bar{a}_{xy})$$

or $c^{y}\bar{A}^{1}_{xy} = c^{x}\bar{A}_{x\overset{1}{y}} + (c^{x} - c^{y})\log_e s\,\bar{a}_{xy}$

or $(c^{x} + c^{y})\bar{A}^{1}_{xy} = c^{x}\bar{A}_{xy} + (c^{x} - c^{y})\log_e s\,\bar{a}_{xy}$

$$= c^{x}\bar{A}_{xy} + (c^{x} - c^{y})\frac{\log_e s}{\delta}(1 - \bar{A}_{xy}).$$

$$= \left\{ c^x - \frac{\log_e s}{\delta}(c^x - c^y) \right\} \bar{A}_{xy} + \frac{\log_e s}{\delta}(c^x - c^y)$$

$$\text{or } \bar{A}^1_{xy} = \left\{ \frac{c^x}{c^x + c^y} - \frac{\log_e s}{\delta} \cdot \frac{c^x - c^y}{c^x + c^y} \right\} \bar{A}_{xy} + \frac{\log_e s}{\delta} \frac{c^x - c^y}{c^x + c^y}$$

But $\bar{A}^1_{xy} = \frac{i}{\delta} A^1_{xy}$ and $\bar{A}_{xy} = \frac{i}{\delta} A_{xy}$ approximately.

Hence substituting and reducing we have

$$A^1_{xy} = \left\{ \frac{c^x}{c^x + c^y} - \frac{\log_e s}{\delta} \cdot \frac{c^x - c^y}{c^x + c^y} \right\} A_{xy} + \frac{\log_e s}{i} \cdot \frac{c^x - c^y}{c^x + c^y}$$

3. *What various forms of investment are now open to life insurance companies? Discuss their relative desirability.*

Railroad Bonds: They yield a fair return, are available in considerable quantity, when well selected furnish ample security, and are readily convertible.

Government and Municipal Bonds: The interest return, except for small issues, is comparatively low, but they are sometimes useful for deposit purposes.

Mortgage Loans on improved property, with a good margin, are a safe investment and yield a return somewhat higher than do first-class bonds.

Stocks: Great care in selection is necessary. Some first-class railroad, bank and trust companies stocks however, furnish a safe and remunerative investment.

Loans on Policies: This is one of the most desirable forms of investment, but only a limited proportion of the funds can be invested in this way.

4. *Show how the reserve on any policy at the end of n years can be expressed in terms of the net premium and the commutation symbols for age at entry and age attained. Hence show how policies of all kinds and durations may be grouped together by attained age for valuation purposes.*

Let ${}_nV$ be the reserve at the end of n years. S the sum assured and π the net premium, then we have

$$\pi\,(N_x - N_{x+n}) = S\,(M_x - M_{x+n}) + {}_nV.\ D_{x+n}$$

$$\text{or } {}_nV = \pi \frac{N_x - N_{x+n}}{D_{x+n}} - S. \frac{M_x - M_{x+n}}{D_{x+n}}$$

This expresses the reserve in terms of the net premium and commutation symbols for age at entry and age attained. This equation can also be expressed in the form

$${}_nV = S.\,A_{x+n} - \pi\,(1 + a_{x+n}) + \frac{\pi N_x - S M_x}{D_{x+n}}$$

$$= S.\ A_{x+n} - \pi\,(1 + a_{x+n}) + \theta . \frac{D_z}{D_{x+n}}$$

where $\theta = \dfrac{\pi\, N_x - S.\ M_x}{D_z}$ and z is some convenient advanced age.

If then the quantities S. π and θ be recorded once for all on each policy a classification may be made by age attained and an account kept of the totals of the three functions for each group. These can be multiplied by the respective valuation functions A_{x+n}, $1 + a_{x+n}$ and $\dfrac{D_z}{D_{x+n}}$ and a summation will give the valuation required. If the conditions of the policy are such that the value of S or of π changes after the policy is in force, a corresponding change in θ will have to be made; the new value of θ being such that at the date when the change takes effect the value on the new set of constants will be the same as on the old.

5. *Under what conditions is the declaration of a simple reversionary bonus of a uniform percentage of the face of the policy for all ages, durations, and kinds of policy, equitable?*

Profits may for practical purposes be divided into two parts, one to be apportioned in proportion to the reserve on the policy and the other in proportion to the effective loading, so that the equitable share of a policy may be expressed in the form

$$a\,l + b\,.\,{}_nV = a\,l + b\,\frac{A_{x+n} - A_x}{1 - A_x}$$

for ordinary life policies; for endowments the corresponding changes in the formula should be made. If the proposed method is to be fair, this should be capable of expression in the form $c\,A_{x+n}$ where c is constant for all values of x and n that is

$$a\,l - b\,\frac{A_x}{1 - A_x} + \frac{b}{1 - A_x}\,.\,A_{x+n} = c\,A_{x+n} \text{ for all values of } x$$

and n. Hence $a\,l - b\,\dfrac{A_x}{1 - A_x} = o$ or $l = \dfrac{b}{a}\,.\,\dfrac{P_x}{d}$ or the effective loading is a constant percentage of the net premium, also $\dfrac{b}{1 - A_x} = c$.

This last condition cannot be fulfilled as A_x is a variable and b and c are constants unless the valuation basis is varied according to age at issue and kind of policy so that b would vary also, being made proportional to the difference between the rate of interest assumed and that actually earned.

6. *Expand* $\dfrac{x}{(1+x)^t - 1}$ *in ascending powers of* x.

Hence derive an approximate formula for the sum of a series when only every n^{th} *value is known.*

See Institute of Actuaries' Text Book, Part 2, Chapter 22, Article 30, and Chapter 24, Article 20, and Transactions of Actuarial Society of America, Volume 8, page 58.

7. *A policy is drawn payable as follows: "unto Jane Doe, beneficiary, wife of the insured," without any further provision. The beneficiary dies and the insured wishes to surrender the policy. Discuss the rights of the insured in such a case.*

A policy so worded would at the death of a wife inure to the benefit of the children, if any, and according to the law of some states could not be assigned. Where an assignment is possible the children should join in the release, either individually, if adult, or if minors, through their general or special guardian. Where no children survive, her interest will pass to her heirs.

8. *A set of select tables are to be graduated in which the mortality in the early policy years at ages below twenty-five at entry is higher in the ungraduated data than at several higher ages. What method would be most satisfactory? Give details of the work of applying the method you would adopt.*

As it is impossible to represent, by a table graduated by Makeham's law unmodified the special feature referred to, that law should not be used unless an extended use is expected to be made of the resulting table in joint life calculations. No perfectly satisfactory method has, however, yet been devised of applying summation graduations, such as those of Woolhouse and Higham, to select tables. The graphic method of graduation would, therefore, be the most suitable. The expectations of life for the separate ages at entry should be made from the ungraduated data. These will indicate what groupings should be made to obtain a series as regular as possible. The data for each group should then be combined and a select table graduated by the graphic method. These tables will indicate where it is possible to merge the select tables into the ultimate, and will furnish a guide for drawing the curves of mortality for each of the early insurance years. A mathematical formula approximating to the facts can generally be used to advantage as a basis, as then only the departure of the facts from the formula remains to be graduated and a larger scale can be adopted.

9. *Discuss the propriety of issuing single premium policies on the participating plan. How would you compute rates for such policies?*

There does not appear to be any conclusive objection to doing so. It is true that one considerable item can be determined more accurately in advance in the case of single premium than in the case of annual premium policies, namely, expense of collecting premiums, also that the lapse feature is practically eliminated. There are other elements, however, which are subject to variation, and these are sufficient to justify issuing the policies on the participating plan.

As any lack of exact justice in the premium rate can be adjusted in the dividends, the loading can be made to conform to the general rule of the company for participating business.

10. *Show what relationship must exist between the values of p_x by two different tables of mortality in order that the policy values by the two tables for the same rate of interest may be the same. Discuss other forms of policy as well as ordinary life.*

We have for any form of policy where π is the net premium and i the rate of interest $({}_{n-1}V + \pi)(1 + i) = {}_nV + q_{[x]+n-1}(S_n - {}_nV)$ where ${}_{n-1}V$ and ${}_nV$ are the terminal reserves for the $(n-1)^{th}$ and n^{th} years respectively, and S_n the sum assured in the n^{th} year. Let $q^1_{[x]+n-1}$ be the rate of mortality in the n^{th} year according to some other mortality table, giving the same terminal reserves but a different net premium π^1. Then

$$({}_{n-1}V + \pi^1)(1 + i) = {}_nV + q^1_{[x]+n-1}(S_n - {}_nV)$$

or, subtracting,

$$(\pi^1 - \pi)(1 + i) = (q^1_{[x]+n-1} - q_{[x]+n-1})(S_n - {}_nV).$$

The left side of the equation is constant so long as the premiums do not change so that the difference in the rates of mortality must be inversely proportional to the net amount at risk. In the case of ordinary life policies two aggregate tables may give the same terminal reserves (see Associate B, question 6), but for other forms one at least of the tables must be in the analyzed or select form. For policies which have become paid-up the left side vanishes and consequently no variation in the rate of mortality is permissible. For endowments in the final year, the formula calls for an infinite difference in the rates of mortality, showing an impossible condition.

11. *Deduce a formula for the distribution of profits on the contribution plan, and state what facts would govern you in determining the constants.*

Under what theory of expense assessment is a dividend earned during the first policy year?

In the case of any policy we have $({}_{n-1}V + \pi)(1 + i) - q_{x+n-1} = (1 - q_{x+n-1})\,{}_nV$ where π is the net premium; i the assumed rate of interest and q_{x+n-1} the tabular rate of mortality. Suppose now the loading is l and the expenses are e, that the actual rate of interest earned is j and the actual rate of mortality is q^1_{x+n-1} then the profits earned during the year are

$$({}_{n-1}V + \pi + l - e)(i + j) - q^1_{x+n-1} - (1 - q^1_{x+n-1})\,{}_nV$$

or substituting from the equation above

$$(l - e)(1 + j) + (j - i)({}_{n-1}V + \pi) + (q_{x+n-1} - q^1_{x+n-1})(1 - {}_nV)$$

The rate of interest earned is determined by dividing the interest earnings, less investment expenses, by the mean funds (reserve and surplus) of the year less one-half the interest earnings. An average rate should be used based on the company's recent experience. The rate of mortality should be based on the company's experience over a sufficient period to give stability. The expenses after deducting off-setting items, such as gain from lapses, give when distributed the expense factor.

If it is assumed that any excess of the cost of new business over the profit from lapses and gain from mortality of recently selected risks is properly assessable on the business at large as necessary for the maintenance of the company, a dividend will be considered as earned the first policy year.

12. *A woman in receipt of an income under an annuity on her own life desires to have the annuity changed so as to be payable so long as either she or her husband survives. What action would you take on the request and what circumstances would you take into consideration?*

In the case of such a request, the company should decline to make the proposed change and suggest that application be made for a survivorship annuity, in favor of the husband. In the consideration of this application the various questions affecting the desirability of the risk could be taken up in the regular way. If it is desired that the annuity be continued for the full amount, such additional premium as is necessary being paid now to the company, the case would be more favorable for action.

13. *Express, in the form of an integral, the value of* $\bar{a}_{xy\overset{1}{z}|w}$ *and lay out a schedule showing how you would apply a formula of approximate integration to its evaluation.*

See Institute of Actuaries' Text Book, Part II, Chapter XV, Art. 29. The arrangement of work given in Article 24, with the necessary additional column for the additional life, would be convenient.

14. *Under-average lives have been assured with extra premiums payable during the continuation of the contract. Under what conditions, if at all, can the extra premium be remitted?*

Where extra premiums have been charged on account of an impairment they should not in general be remitted. The fact of the assured subsequently passing a good medical examination is not sufficient ground for such action. If, however, other circumstances are such as to indicate clearly that an error was made by the medical examiner originally, the extra may be remitted. Also if the business has been transacted under such conditions that, at the time the application is made, it would be more profitable to the company to remit the extra and retain the healthy life on those terms than to refuse and cause a lapse, prudence would dictate its remission. This could in general only occur during the first policy year.

15. *Show how Woolhouse's formula for graduation was obtained and draw up a schedule of operations by which it can be most conveniently applied.*

Woolhouse's formula for graduation was derived by considering the five different series, which can be formed by taking values from the original series separated by quinquennial intervals. Each series is then supposed to be filled out by central difference interpolation to the second order, and a final series formed by taking the averages of the corresponding values in these five series. We then see that expressing by u^1 the graduated value we have.

$$u^1_x = \tfrac{1}{5}\Big\{\left(-\tfrac{3}{25}u_{x-7} + \tfrac{21}{25}u_{x-2} + \tfrac{7}{25}u_{x+3}\right) + \left(-\tfrac{2}{25}u_{x-6} + \tfrac{24}{25}u_{x-1} + \tfrac{3}{25}u_{x+4}\right) + u_x + \left(\tfrac{3}{25}u_{x-4} + \tfrac{24}{25}u_{x+1} - \tfrac{2}{25}u_{x+6}\right) + \left(\tfrac{7}{25}u_{x-3} + \tfrac{21}{25}u_{x+2} - \tfrac{3}{25}u_{x+7}\right)\Big\}$$

or $125\,u^1_x = -3\,u_{x-7} - 2\,u_{x-6} + 3\,u_{x-4} + 7\,u_{x-3} + 21\,u_{x-2} + 24\,u_{x-1} + 25\,u_x + 24\,u_{x+1} + 21\,u_{x+2} + 7\,u_{x+3} + 3\,u_{x+4} - 2\,u_{x+6} - 3\,u_{x+7}$

Various methods have beeen suggested to simplify the calculations involved, but probably the simplest is that of J. A. Higham, which consists in arranging the values in columnar form, summing in threes and subtracting three times the result from ten times the central original value, then summing the result in fives three times successively and dividing by 125. See J. I. A. Vol. 31, p. 323.

16. *Gross premiums are to be computed, which, after covering certain initial and renewal expenses, which are partly proportional to the premium and partly constant, will provide, on the assumption of four per cent interest, a simple reversionary bonus of one per cent per annum. Give a formula for the premium and state what basis of valuation should be adopted.*

Let π^1_x be the required office premium, and suppose the initial expenses are $k\,\pi^1_x + a$ per unit assured and the renewal expense $l\,\pi^1_x + b$. Also suppose, to be perfectly general, the policy in question s an endowment, with a period of n years and with premiums for m years. We have then where all monetary values are on a four per cent. basis

$$\pi^1_x(1 + a_{x\,\overline{m-1}|}) = A_{x\,\overline{n}|} + .01\,\frac{R_{x+1} - R_{x+n+1} + n\,(D_{x+n} - M_{x+n})}{D_x} + k\,\pi^1_x + a + l\,\pi^1_x\,a_{x\,\overline{m-1}|} + b\,a_{x\,\overline{n-1}|}$$

$$\text{or } \pi^1_x = \frac{A_{x\,\overline{n}|} + a + b\,a_{x\,\overline{n-1}|} + .01\,\dfrac{R_{x+1} - R_{x+n+1} + n\,(D_{x+n} - M_{x+n})}{D_x}}{(1-k) + (1-l)\,a_{x\,\overline{m-1}|}}$$

A net valuation on a three and one half per cent. basis, with a special reservation for expenses on limited payment policies after they have become paid-up, would probably on an average distribution of the business be sufficient to maintain the dividends.

FELLOW.

17. *Policies with semi-annual or quarterly premiums are usually valued on the assumption that the annual premium has been paid and credit is allowed in the assets for the gross deferred premium, less loading. How does the reserve thus obtained compare with that obtained by valuing the policies as strictly half-yearly or quarterly contracts?*

See Institute of Actuaries Text Book, Part II, Chapter XVIII, Articles 84 to 88.

It is to be noted in this connection that, in this country where premiums are payable semi-annually or quarterly they are merely instalments of the annual premium and not true semi-annual or quarterly premiums; as any unpaid balance is deducted from the death claim.

18. (*a*) *What are the three principal sources of surplus and in what proportions do they generally contribute?*

(*b*) *How would you treat investment expenses in determining the share of profits from each source?*

(*a*) The question of the proportions in which the various sources contribute to profits is one to which various answers may be given, according to the point of view. When we remember, however, that what appears as gain from surrenders and lapses, is in large part merely a refund of initial expenses not yet reimbursed out of loadings and that the bulk of what appears as gain from interest is really interest earned on surplus, the distribution of which is deferred, it would appear that the main sources of profit, at the present time, in the order of their importance, are gain from excess of loading over net expenses, mortality savings and interest earned in excess of the rate necessary to maintain the reserve. The two former probably contribute about equally, and the last considerably less than either of them.

(*b*) Investment expenses which are necessary for the care of old investments, including changes of securities and reinvestments, should be considered as a charge against interest income.

19. (*a*) *A company grants, as surrender values under its policies, cash, paid-up policies or extended insurance. How would you fix the relative amounts of these surrender values?*

(*b*) *What theoretical objections are there to extended insurance in case of endowments?*

(a) The amount of the paid-up policy to be given, should be determined by deducting from the reserve on the policy a charge for initial expenses not yet reimbursed. The manner of determining the amount of this charge should vary with the practice of the company regarding dividends and other features. The balance after deducting this charge should then be converted at net rates into paid-up assurance. Theoretically, a further charge should be made on account of the justifiable presumption that a person surrendering his policy is a select life, but, so far as paid-up policies are concerned, this is already sufficiently provided for, if the reserves used as a basis are based, for premiums as well as for other factors, on an ultimate table. In converting into cash, however, account should ordinarily be taken of this possible selection. And in

determining the period of extended insurance, a sufficient loading should be used to compensate for the selection adverse to the company, which may be exercised by the assured.

(*b*) Applications are frequently accepted on the endowment form which would not be accepted for term assurance, the expectation being that near the end of the period the amount at risk would be small and that the effect of extra mortality occurring at that time might be neglected. If however extended assurance is given after a few years' premiums have been paid the company would be on the risk for the full amount as the reserve would be small.

20. *Find an approximate expression for* $\mathring{a}_x$ *and show how it must be modified to give the value of* $\mathring{a}_x^{(m)}$.

See Institute of Actuaries Text Book, Part II, Chapter XI, Articles 5 and 8.

21. *What are the legal requirements as to basis of valuation in order that new business may be transacted?*

To what point must the reserve be reduced before a receivership can be insisted on?

In the State of New York, a company is not permitted to transact new business, if a statement of assets and liabilities, including the value of its policies on the basis of the Actuaries' or Combined Experience Table with four per cent. interest, for policies issued prior to January 1st, 1901 and on the basis of the American Experience Table, with three and a half per cent. interest, for policies issued since that date or on such higher basis as the company may have adopted, shows an impairment of more than fifty per cent. of the capital. But a receiver cannot be appointed unless the funds invested according to law, after deducting the outstanding liabilities and capital stock, fall below the reserve on the basis of the American Experience Table with four and one half per cent. interest.

See section 82 of the general Insurance Law.

22. (*a*) *State Gompertz's hypothesis as to the law of human mortality and deduce the value of* l_x *in terms of the constants.*

(*b*) *What modification was introduced by Makeham, and how did it affect the value of* lx*?*

See Institute of Actuaries' Text Book, Part II, Chapter VI, Articles 9, 10, 14 and 15.

23. (*a*) *Give a brief outline of the usual form of statement required of a life company, enumerating the various schedules.*

(*b*) *In preparing the annual statement a company wishes to write off two per cent of the value of its real estate. Show two ways in which this may be done.*

(*a*) See standard blank for returns of life companies.

(*b*) Any desired amount may be written off the value of real estate, either by passing the amount through disbursements under a special profit and loss item and correspondingly reducing the item of

book value of real estate under assets, or by setting up an item in the liabilities of a real estate depreciation fund.

24. *Give the formula for the reserve on a survivorship annuity with return of premiums in case of prior death of the nominee. What basis would you adopt for valuation?*

Let (x) be the assured and (y) the nominee, and let π be the net premium and π^1 the gross premiun per unit; then

$$\pi(1 + a_{xy}) = a_y - a_{xy} + \pi^1(IA)^{\ 1}_{xy} \quad \text{or} \quad 1 + \pi = \frac{1 + a_y + \pi^1 \dfrac{R^{\ 1}_{xy}}{D_{xy}}}{1 + a_{xy}}$$

Then the reserve at the end of n years will be

$$a_{y+n} - a_{x+n:y+n} + \pi^1 \left(nA_{\overline{x+n}:\overset{1}{\overline{y+n}}} + (IA)_{\overline{x+n}:\overset{1}{\overline{y+n}}} \right)$$

$$- \pi (1 + a_{x+n:y+n}) = (1 + a_{y+n}) - (1 + \pi)(1 + a_{x+n:y+n})$$

$$+ \pi^1 \frac{nM_{\overline{x+n}:\overset{1}{\overline{y+n}}} + R_{\overline{x+n}:\overset{1}{\overline{y+n}}}}{D_{x+n:y+n}}$$

Theoretically, an assurance table should be used for the assured and an annuity table for the nominee, but in practice the labor would be prohibitive and the Carlisle table, besides being convenient, provides for a sufficient reserve. The same rate of interest should be used as for other annuities. Three and one half per cent. would be sufficiently conservative.

25. *What will be the probable effect upon the dividends to the survivors of an extra mortality in the eighteenth to the twentieth years under twenty-year endowment policies issued on the twenty-year dividend plan? State the reasons for your answer.*

Each extra death claim reduces the total profits to be divided, by the amount of premiums unpaid less expenses and by the interest on those premiums and on the amount of the claim. If this amount is less than the share of profits which would have been allotted to the policy at maturity, had the policyholder survived to that time, the reduction in the number sharing in the profits will more than counterbalance the reduction in the total profits and the share of each survivor will consequently be increased. In the case of twenty year endowment policies on the deferred dividend plan this would generally be the case after the eighteenth year.

26. (*a*) *A twenty-year endowment has been in force five years and it is desired to change it to an ordinary life policy. What allowance should be made for the higher premium paid in the past?*

(*b*) *What difference would you make in such cases as between annual dividend and deferred dividend policies?*

(*a*) In making a change such as this the maximum allowance which can be made is the difference in reserves on the two policies with such deduction as may be necessary on account of the unrecouped initial expenses. The amount allowed should, of course,

not exceed the difference between the surrender values of the two policies, and a rule to allow that difference would in practice produce fair results.

(*b*) The rule in this form would apply to either annual or deferred dividend policies the appropriate difference in treatment being provided for by the difference, if any, in values allowed on the two forms of contract.

27. *Evaluate the following integrals:*

$$\text{(a)} \int_0^\infty x^3 e^{-x}\, dx.$$

$$\text{(b)} \int_a^b (x-a)^3 (b-x)^2\, dx.$$

$$\text{(c)} \int_0^a \frac{x^2 + 3x + 1}{x+1}\, dx.$$

$$\begin{aligned}
\text{(a)} \int_0^\infty x^3 e^{-x}\, dx &= 3\int_0^\infty x^2 e^{-x}\, dx \\
&= 6\int_0^\infty x\, e^{-x}\, dx \\
&= 6\int_0^\infty e^{-x}\, dx \\
&= 6.
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \int_a^b (x-a)^3 (b-x)^2\, dx &= \tfrac{2}{4}\int_a^b (x-a)^4 (b-x)\, dx \\
&= \tfrac{1}{10}\int_a^b (x-a)^5\, dx \\
&= \tfrac{1}{60}(b-a)^6.
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \int_0^a \frac{x^2 + 3x + 1}{x+1}\, dx &= \int_0^a (x+2)\, dx - \int_0^a \frac{1}{x+1}\, dx \\
&= \frac{a^2}{2} + 2a - \log_e (1+a).
\end{aligned}$$

28. *It is found from a large number of observations that the annual death rate from accident among railroad engineers averages about five per thousand. State under what terms you would grant an engineer a twenty-payment life policy.*

The addition of a little under .005 to the force of mortality at each age is equivalent, so far as annuity values are concerned, to adding one half per cent. to the rate of interest. Expressing, then, net premiums in terms of annuity values, we see that the extra mortality is provided for on ordinary life policies by taking the net premium at a rate of interest one half per cent. higher as a basis

and adding the difference in the rates of discount. And that the net premium for a twenty payment life policy bears the same ratio to that for an ordinary life policy as holds for the corresponding premiums for normal mortality at the higher rate of interest. In other words, the rule for a twenty payment life policy would be to take the net premium at a rate one half of one per cent. higher and increase it in the proportion which the difference in the rates of discount bears to the net ordinary life premium at the higher rate. This provides for an addition to the force of mortality equal to the difference in the force of discount. Now the normal rate of death from accident is about one per thousand so that the extra force of mortality for engineers, according to the data, would be about .004 or eighty per cent. of that which the above rule provides for. This could be adjusted by reducing the extra in that proportion.

THE FUNDAMENTAL PRINCIPLES OF PROBABILITY

I. The Measurement of Probabilities

As mathematical science has to do with quantities and their relations, it is evident that before anything can be done in the way of developing a mathematical theory of probability, we must establish some method of determining the numerical measure of a given probability. It is true that without doing so it is possible to lay down rules of operation and deduce their necessary consequences just as we may say that the area of a rectangle is equal to the product of its length by its breadth even though we do not know and have no means of ascertaining either of those factors. So also, in probability, we are able to deduce certain relations among connected probabilities which hold, even though the exact values of these probabilities are unknown. But in order to determine these relations it is necessary to know the nature of the quantities with which we are dealing and their relation to the entities of which they are the measure.

Probability then has to do with events or things which, in detail and as to the individual items, do not, at some specified time, fall within our knowledge. In so far as anything is a matter of knowledge it is necessarily taken out of the region where probability holds sway.

Again probability has to do with the frequency with which an event happens or a specified result is obtained.

Other things being equal that event which happens the more frequently is the more probable. In fact the numerical measure which has been universally adopted for the probability of an event under given circumstances is the ultimate value, as the number of cases is indefinitely increased, of the ratio of the number of times the event happens under those circumstances to the total possible number of times. From this statement it follows that the greatest possible value of a probability is unity and that this expresses the probability of an event which happens on every possible occasion, or in other words, which is certain to occur. The lowest value is zero which expresses the probability of an event which never occurs. We also see that those events, or ways of happening of an event are equally likely or have equal probabilities which happen in the long run in the same percentage of the possible cases. An individual is said to be selected at random from a group when all the individuals composing the group are equally likely to be selected.

It is stated above that the measure there given has been universally adopted and this holds true in spite of the fact that the rule has been stated in ways which on their face differ widely from that above given. The one most commonly given is that if an event can happen in a ways and fail in b ways all of which are equally likely, the probability of the event is the ratio of a to the sum of a and b. It is readily seen that if we read into this statement the meaning of the words "equally likely," this measure, so far as it goes, reduces to a particular case of that given above. This form of stating the rule is useful in those cases where the happening or failure of the event can be analyzed into a definite number of elementary ways of happening, all of which we may, either on *a priori* grounds or by the conditions of the problem, assume to be equally likely. For example, where three balls are drawn at random from an urn containing

seven white and three black balls, all the different combinations three at a time which are possible, are equally likely as we can go from any one combination to any other by a series of interchanges or substitutions of one ball for another each of which by the condition of the problem that the drawing is made at random leaves the probability unchanged. In order then to determine the probability that the three balls drawn should fulfill any given condition, for instance should be all white, we count the number of combinations which do fulfill it, in this case 35, and the total number 120, and the ratio of these two numbers gives the required probability. It is for this reason that so many problems in the theory of probability reduce to an enumeration of combinations, permutations or arrangements. But the problem in every case under all its various shapes and disguises is the same, namely, to determine in what proportion of the possible cases the event, in the long run, tends to happen. The nature of this tendency will be discussed in the note on repeated trials.

II. The Combination of Probabilities

The next step in the development of the mathematical theory of probability is to determine under what conditions the fundamental operations of addition and multiplication, with their inverses, subtraction and division, give, when performed between probabilities, interpretable results and to ascertain the interpretation of those results.

In order that two probabilities may be added together, it is necessary that they should relate to the same general subject. For example, the result of adding together the chance of throwing six with a die and that of a man aged 35 dying within a year would not be capable of interpretation as a probability. Again the two proba-

bilities must not have any part common to both, as otherwise that part would be included twice in the sum and the result again could not be interpreted.

We thus see that, in order that their probabilities may be added together, two events must be mutually exclusive, that is, it must be impossible for the two events to happen together. When this is the case the sum of the probabilities gives the chance that one or other of the events will happen. This may be seen from the fact that in a large number of trials, since the two events never happen together, the number of times when either one or the other happens is found by adding together the number of times when they respectively happen. For example, consider the chance of throwing more than four with one throw of an ordinary six-faced die. This can be done either by throwing five or by throwing six, the probability of each being one-sixth. Then in the long run for each six trials five will be thrown once and six once or more than four will be thrown twice, and its probability is therefore one-third.

Again, if an urn contains five white, seven red and eight black balls and one be drawn at random the chance of drawing a white one is five-twentieths, that of drawing a red one is seven-twentieths and the chance of drawing either a white or a red ball is the sum of these two, or twelve-twentieths, as may be readily seen from the fact that twelve of the twenty balls are either white or red.

From the above reasoning it follows since subtraction is the inverse of addition that the probability to be subtracted must be included in and form a part of the probability from which it is to be subtracted. For example, in the case above cited, if we know that the probability of either a white or red ball being drawn is three-fifths and that the probability of a white ball being drawn is

one-fourth, then, by subtraction, the probability of a red ball is the difference or seven-twentieths.

A prominent example of the subtraction of probabilities arises in connection with the fact that since an event must either happen or fail the chance of one or the other result is unity. So that by subtracting from unity the probability of an event happening we get the probability of its failing. These two probabilities are said to be complementary to one another.

Since a probability is a fraction of which the denominator is the number of times an event can possibly happen, and the numerator is the number of times it does happen, it follows that if the product of two probabilities is to be itself a probability they must be so connected that the denominator of one represents the same class of events as the numerator of the other. In other words, if one factor is the probability of an event happening, the second factor must be the probability of some other event also happening if the first does, and in that case the product represents the probability of both events happening together. For example, suppose an urn contains one hundred balls of which twenty are ivory and that, of these twenty, five are white. Then, if one ball be drawn at random the chance that it is ivory is one-fifth, and the chance that if it is ivory it will be white is one-fourth, and we see, from the fact that of the total of one hundred balls five are white ivory ones, that the probability of a white ivory ball being drawn is one-twentieth which is the product of one-fourth and one-fifth. Similarly, if at a given age one person, on an average, in each 125 dies within one year and of those who die at that age one in eight on an average die from accident, then in the long run out of each thousand persons eight will die within the year and of these eight one will die from accident. Or the probability of a person dying from accident within one year is

one one-thousandth or the product of one hundred and twenty-fifth and one-eighth.

We have said that one of the factors is the probability that if one event happens another event will also happen. This probability may be different from the probability that if the first event fails the second will happen or it may be the same. In the former case, the events are said to be dependent and it is necessary to attend to the distinction above noted. In the latter case, the events are said to be independent and, as the probability to be used in the multiplication is, in this case, equal to the general probability that the second event shall happen, it is unnecessary to attend to the distinction. This gives the rule that to find the chance that both of two independent events shall happen, we multiply together their respective probabilities.

The rule for independent events is the one which is most frequently used in practice, because in a great many cases, we may, in the absence of any reason to doubt it, assume that the two events are independent. This, however, does not alter the fact that where two probabilities are multiplied together to give a resultant probability one of the factors is always, either expressly or by implication, the probability that if one event happens another will happen also.

Conversely the probability of a compound event may be divided by the probability of one of its component simple events, the happening of which is necessary to that of the compound event, and the quotient gives the probability that if the simple event happens the remainder of the compound event will also happen. Or this latter probability may be known and be used as the divisor and the quotient would give the probability of the simple event. For example, suppose we know that at a given age the probability of death from accident within one year is one per

thousand and that of the total deaths at that age one in ten is by accident, then by division we find the total chance of death within one year to be ten per thousand.

All the computations of probabilities are made by the application direct or indirect of these four rules, and most of the errors that arise in such computations come from neglect to observe the restrictions and conditions under which these various operations are permissible. The following is a good example of the combination in one computation of the various elementary rules.

A number of urns are filled with balls. In each of one-tenth of the total number of urns nine-tenths of the balls are white and in each of the remainder four-tenths are white. From an urn taken at random a ball is drawn at random, found to be white and returned. What is the chance that a second random drawing from the same urn would give a white ball?

Here the *a priori* probability of selecting an urn in which nine-tenths of the balls are white is one-tenth and, consequently the chance that such an urn will be selected and a white ball drawn is by multiplication nine one-hundredths. Also the chance of selecting one of the others is the complement of one-tenth or nine-tenths, and consequently the chance that such an urn will be selected and a white ball drawn is, by multiplication, thirty-six one hundredths. Hence, by addition the total chance of a white ball being drawn is forty-five one hundredths. From this we see by division that if a white ball is drawn the chance that it is drawn from one of the first set of urns is nine forty-fifths or one-fifth and the chance that it is drawn from one of the second set is four-fifths. If then a second drawing is made from the same urn the chance that the urn will belong to the first set and that a white ball will be drawn is nine-tenths of one-fifth or nine-fiftieths, and the chance that the urn will belong to the second set and that a white ball will

be drawn is four-tenths of four-fifths or sixteen-fiftieths. Hence the total probability of a white ball being drawn is twenty-five fiftieths or one-half.

III. Expectation or Mean Values

Frequently in applications of the theory of probability the important element in the problem is the value assumed, in the various contingencies, by some variable quantity. This quantity may be the number of successes in a given number of trials, the distance of a point from another point, a line or a plane, the duration of life, the amount of money to be paid or received or the present value of such amount or any other quantity connected with the result of the trial, including any function explicit or implicit of any of these quantities. In these cases the special point to which attention is generally directed is the mean value of the variable quantity, or the ultimate value, as the number of trials is indefinitely increased, of the average of the values of the variable resulting from the various trials. In certain cases this function is also called the expectation from each trial, the idea being that a certain aggregate result is expected from a large number of trials, and that, by dividing this aggregate result by the number of trials, we get the expectation from each. It is evident that this average will be determined by multiplying each value of the variable by the number of times it occurs and dividing the sum of the products by the total number of trials. In the ultimate limit this is equivalent, from the definition of probabilities, to multiplying each value of the variable by the probability of its occurrence and adding all the products together. For example, suppose one dollar is to be received if a coin, about to be tossed, should turn up head. In a large number of such trials the net results would be that one dollar would be received for every two trials or

the expectation from one trial is one-half of one dollar, or one dollar multiplied by the chance of head being thrown.

It follows from the way in which the mean value is arrived at, that the mean value of the sum of the values of the variable resulting from a specified number of trials or the expectation from such trials may be obtained by multiplying the expectation from each single trial by the number of trials. Thus the expectation of success in one trial is equal to the probability and the expected number of successes in a given number of trials is the product of that number into the probability.

An important consideration arises in connection with the mean value of the product of two independent variables, that is, of the product of two quantities whose variations depend on contingencies which are independent of each other. In the computation of this mean value all the possible values of the product and their respective probabilities will be represented by the various terms in the expression for the product of the mean values of the individual variables, so that the mean value of the product is equal to the product of the mean values. As an example, suppose a sum of money is to be received which is to be determined, by tossing a penny to decide whether payment will be in dimes or in nickels, and by throwing a die to determine the number of coins, the number to be equal to the number thrown. Then the mean value of each coin is seven and one-half cents, and the mean number of coins is three and one-half and the expectation therefore twenty-six and a quarter cents, as may also be seen by directly taking the mean of the amounts payable in the twelve equally likely possible cases.

It is to be carefully noted that this rule, that the mean value of the product is equal to the product of the mean values, applies only when the two quantities involved are entirely independent. Where the variation of one quantity

is in any way dependant on that of the other, this relation cannot be depended upon. As a case in point, the mean value of the square of a variable quantity is always greater than the square of the mean value. This appears from general considerations, but it may be also demonstrated algebraically and the difference may be shown to be equal to the mean value of the square of the difference between the actual value of the variable and its mean value. For example, in one trial the mean value of the square of the number of successes is equal to the probability of success since it is equal to unity multiplied by that probability added to zero, multiplied by the complementary probability. We have already seen that the mean value of the number of successes is also equal to the same probability. The mean value of the square of the departure of the actual from the expected is seen by a direct computation to be equal to the product of the probability into its complement, or to the difference between the probability and its square, which is in agreement with the statement made above.

IV. Repeated Trials

Where a number of trials are made the chance that the event should happen every time is, by the principle of the multiplication of probabilities, equal to the probability of success at a single trial raised to a power equal to the number of trials. Similarly where p is the probability of success at a single trial and q the complementary probability of failure, the chance that of n trials, r particular ones should be successful and the remainder unsuccessful, is determined by multiplying p raised to the rth power, by q raised to a power equal to the difference between n and r. We thus see that the probability of the success at any particular r trials only, is equal to the probability for any other particular r trials only, and that consequently the total prob-

ability of exactly r successes is found by multiplying this probability by the number of combinations of n things r at a time. Accordingly the ratio of the chance of exactly r successes to that of exactly $r-1$ is compounded of the ratio of the chance of success on a particular r occasions only to that of success on a particular $r-1$ occasions only, which is equal to the ratio of p to q, and the ratio of the number of combinations of n things r at a time to the number $r-1$ at a time, which is equal to the ratio of $n+1-r$ to r. This ratio will be greater than unity or the chance of exactly r successes will be greater than that of exactly $r-1$, if $p(n+1-r)$ is greater than qr or if r is less than $p(n+1)$. We thus see that the most probable number of successes is determined by taking the integral part of the expected number of successes in $n+1$ trials and that the probability increases regularly, without any retrograde movement up to the maximum and then decreases in the same way.

We may have r successes out of $n+1$ trials in two ways only, by $r-1$ successes in n trials followed by success in the other or by r successes followed by failure, its probability therefore is intermediate in value between those of $r-1$ and r successes in n trials. We thus see that the probability of the most probable number of successes in $n-1$ trials is in general less than the corresponding probability for n trials as it must be less than the greater of the probabilities of some two consecutive numbers of successes unless their two probabilities are equal, in which case it will be equal to either.

We thus see that as the number of trials increases the maximum value of the probability of an exact number of successes tends to diminish sometimes halting but never retrograding. It follows that the chance, that the actual number of successes will be within any definite number of the expected, decreases as the number of trials is increased. This result appears at first sight inconsistent with the funda-

mental definition of the measure of probability, but it is not in fact so inconsistent as the definition refers to the ratio of the number of successes to the number of trials and this relates only to the number of successes. We shall proceed to investigate the variations of the ratio of successes.

We have seen in the note on mean values that the mean value of the square of the departure of the actual from the expected number of successes in one trial is equal to $p\ q$. Also the departure of the actual from the expected in n trials is equal to the algebraic sum of the departure in the individual trials, so that the square of the departure in n trials is equal to the sum of the squares of the departures in the individual trials together with twice the sum of their products two and two. But these departures for the individual trials are independent of one another and their mean values are each zero, consequently by the principle established in the note on Mean Values, so also is the mean value if the product of any pair of them. We thus see that the mean value of the square of the departure of the actual number of successes from the expected number in n trials is n times the value for one trial or is equal to $n\,p\,q$. But the departure of the actual proportion of successes from the expected is found by dividing the departure of the number by n the number of trials, so that the mean value of its square is determined by dividing $n\,p\,q$ by the square of n or by dividing $p\,q$ by n. This quotient is seen to diminish as n increases.

Let now h be an assigned departure and let P be the probability that the actual proportion of successes differs from the expected by more than h. Then it is evident that the mean value of the square of the departure is not less than P times the square of h. Or in other words n P is not greater than the quotient of $p\,q$ by the square of h. It follows then that no matter how small h is, provided it does not vanish, we can by taking n large enough make P

less than any assigned quantity. We thus see that as n increases the actual proportion becomes more and more nearly certain to agree within narrower and still narrower limits with the expected proportion. This is the nature of the tendency to the ultimate value as the number of trials is indefinitely increased.

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